For a $C^2$-function $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ and a nonnegative integer $n \in \mathbb{Z}_{\geq 0}$, let’s define

$$I_n(f) = \int_0^{\pi/2} f(t) \cos^{2n} t \, dt.$$ 

Now by the integration by parts, we obtain

$$I_n(f'') = [f'(t) \cos^{2n} t]_0^{\pi/2} + \int_0^{\pi/2} f'(t)(2n \cos^{2n-1} t \sin t) \, dt$$

$$= -f'(0) + 2n(2n-1)I_{n-1}(f) - 4n^2 I_n(f)$$

if $n \in \mathbb{Z}_{\geq 1}$. In particular, by setting $f(t) = \cos(2tx)$ we have the following

$$2(n^2 - x^2)I_n(\cos(2tx)) = n(2n-1)I_{n-1}(\cos(2tx)). \quad (1)$$

As a special case, we have the well-known recursive formula

$$2n^2 I_n(1) = n(2n-1)I_{n-1}(1). \quad (2)$$

By dividing these two identities, we obtain

$$\frac{I_n(\cos(2tx))}{I_n(1)} \left(1 - \frac{x^2}{n^2}\right) = \frac{I_{n-1}(\cos(2tx))}{I_{n-1}(1)}.$$ 

So we have

$$\frac{I_n(\cos(2tx))}{I_n(1)} \prod_{k=1}^{n} \left(1 - \frac{x^2}{k^2}\right) = \frac{I_0(\cos(2tx))}{I_0(1)} = \frac{\sin(\pi x)}{\pi x}.$$ 

(Of course if $x = 0$, we should interpret the right hand side as 1.)

Note that

$$I_n(1) - I_n(\cos(2tx)) = I_n(2\sin^2(tx)) \geq 0.$$ 

Also note that for $0 \leq t \leq \frac{\pi}{2}$, we have $t \leq \frac{\pi}{2} \sin t$. Therefore we get

$$0 \leq |\sin(tx)| \leq |tx| \leq \frac{\pi \sin t}{2}|x|.$$ 

Hence, we have the following inequality

$$I_n(1) - I_n(\cos(2tx)) \leq \frac{\pi^2 x^2}{2} I_n(\sin^2 t) = \frac{\pi^2 x^2}{2}(I_n(1) - I_{n+1}(1)).$$

Therefore we have

$$0 \leq 1 - \frac{I_n(\cos(2tx))}{I_n(1)} \leq \frac{\pi^2 x^2}{2} \left(1 - \frac{I_{n+1}(1)}{I_n(1)}\right) = \frac{\pi^2 x^2}{2} \left(1 - \frac{2n+1}{2n+2}\right)$$

and it leads to our conclusion. Remark that the last equality comes from (2).